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# Evolution towards an invariant density distribution under a discrete-time quadratic map

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**Abstract.** Recently, Nandakumaran has proved that under the discrete-time quadratic map $x_{t+1}=4x_t(1-x_t)$, all the probability density distributions

$$r_0^{(n)}(x)=\frac{x^n(1-x)^n}{B(n+1,\, n+1)} \qquad 0\leqslant x\leqslant 1 \qquad n=0,1,2,\ldots$$

converge towards an invariant limit density associated with the map when $t$ tends to infinity. The purpose of the present paper is to generalise this result. Starting from the Fourier series expansion of any real single-valued function which satisfies the Dirichlet conditions in $[0, \pi/2]$, a broad class of normalised initial functions $\{w_0(x)\,|\,0\leqslant x\leqslant 1\}$ is obtained, each of which is the sum of a convergent series involving the Chebyshev polynomials of both kinds. The evolution equation for this class of functions under the map mentioned is found explicitly. It is shown that the absolute convergence of the series expansion of the symmetric part in an initial $w_0$ function is a sufficient condition to ensure an evolution for $t\to+\infty$ towards the same invariant limit density as obtained by Nandakumaran with the infinite set of particular initial densities $r_0^{(n)}(x)$. Some auxiliary results related to the treatment of the considered problem are also presented.

In a recent letter, Nandakumaran (1985) studied the evolution of the probability density distribution

$$r_0^{(n)}(x)=\frac{x^n(1-x)^n}{B(n+1,\, n+1)} \qquad \forall x\in[0,1] \qquad \forall n\in\mathbb{N} \tag{1}$$

under the discrete-time quadratic map

$$x_{t+1}=4x_t(1-x_t) \tag{2}$$

and showed that it converges towards the invariant limit density

$$r(x)=\frac{1}{\pi[x(1-x)]^{1/2}} \qquad \forall x\in[0,1] \tag{3}$$

for every value of $n$, thus generalising the result obtained by Falk (1984) in the special case $n=0$. Formulated more explicitly, one constructs the sequence $\{r_t^{(n)}(x)\,|\,t\in\mathbb{N}_0\}$ generated by

$$r_{t+1}^{(n)}(x)=\frac{1}{4(1-x)^{1/2}}\left[r_t^{(n)}\left(\frac{1-(1-x)^{1/2}}{2}\right)+r_t^{(n)}\left(\frac{1+(1-x)^{1/2}}{2}\right)\right] \qquad \forall t\in\mathbb{N} \tag{4}$$

starting from (1) and it appears that, for any $n\in\mathbb{N}$,

$$\lim_{t\to+\infty} r_t^{(n)}(x)=\frac{1}{\pi[x(1-x)]^{1/2}}. \tag{5}$$

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Since the algorithm (4) is linear and homogeneous with respect to $r$ and the limit density (3) is independent of $n$, it is clear that any probability density distribution which is a polynomial in integer powers of $x(1-x)$, i.e.

$$R_0(x)=\sum_{n=0}^{\nu} c_n \frac{x^n(1-x)^n}{B(n+1,n+1)} \qquad \sum_{n=0}^{\nu} c_n=1 \qquad \nu\in\mathbb{N}_0 \qquad \forall x\in[0,1] \tag{6}$$

also approaches (3) under the transformation (4). This immediately raises the question: is any normalised initial function $w_0(x)$ defined on $[0, 1]$ and possibly subjected to certain conditions, also endowed with the property that, under the transformation (4), it approaches (3)? In this paper, a broad class of functions having this property will be obtained.

It is well known that if a real single-valued function $f(\varphi)$ defined on $[-\pi, \pi]$ satisfies the conditions of Dirichlet, it can be expanded in a unique way into a Fourier series in terms of cos $n\varphi$ and sin $n\varphi$ with integer $n$ values. If one maps $[-\pi, \pi]$ linearly onto $[0, \pi/2]$, one arrives at

$$f(4\theta-\pi)\equiv g(\theta)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos 4n\theta+b_n\sin 4n\theta) \tag{7a}$$

with

$$a_n=\frac{4}{\pi}\int_0^{\pi/2} g(\theta)\cos 4n\theta\,\mathrm{d}\theta \qquad b_n=\frac{4}{\pi}\int_0^{\pi/2} g(\theta)\sin 4n\theta\,\mathrm{d}\theta. \tag{7b}$$

Next, setting $x=(\sin\theta)^2$, with $\theta=\sin^{-1}(x^{1/2})$ as inverse, $x$ being restricted to $[0, 1]$, one obtains

$$g(\sin^{-1}x^{1/2})\equiv w_0(x)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}[a_nT_{2n}(1-2x)+2b_nx^{1/2}(1-x)^{1/2}U_{2n-1}(1-2x)] \tag{8a}$$

with

$$a_n=\frac{2}{\pi}\int_0^1 w_0(x)\frac{T_{2n}(1-2x)}{x^{1/2}(1-x)^{1/2}}\,\mathrm{d}x \qquad b_n=\frac{4}{\pi}\int_0^1 w_0(x)U_{2n-1}(1-2x)\,\mathrm{d}x \tag{8b}$$

where $T$ and $U$ are the familiar symbols for the Chebyshev polynomials of the first and the second kind, respectively. Note that the right-hand side of (8$a$) may be transformed in such a manner that $w_0(x)$ is expressed solely in terms of functions depending on $x$ through the argument $x(1-x)$:

$$w_0(x)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_nT_n[1-8x(1-x)] \pm 4b_n\{x(1-x)[1-4x(1-x)]\}^{1/2}U_{n-1}[1-8x(1-x)]) \tag{9}$$

in which the + sign refers to $0\leq x\leq\frac{1}{2}$ whereas the − sign holds for $\frac{1}{2}<x\leq 1$. If the real single-valued function $w_0(x)$ defined on $[0, 1]$ is bounded and continuous, possibly except at a finite number of abscissae $x_1, x_2, \ldots, x_q$ in $]0, 1[$ where it exhibits finite jumps, if at any such point $x_p$

$$w_0(x_p)=\tfrac{1}{2}[w_0(x_p-0)+w_0(x_p+0)] \tag{10a}$$

if

$$w_0(0)=w_0(1)=\tfrac{1}{2}[w_0(1-0)+w_0(+0)] \tag{10b}$$

and if $[0, 1]$ can be subdivided into a finite number of open subintervals in each of which $w_0(x)$ is monotonic, *lato sensu*, then the series in (9) which is deduced from (7*a*) is convergent for any $x \in [0, 1]$, its sum is $w_0(x)$ and the convergence is uniform in any subinterval of $[0, 1]$ not containing a discontinuity of $w_0(x)$ or having such an abscissa as a bound. Note that the series constructed with the $a$ coefficients and the $b$ coefficients separately are also convergent and that

$$\tfrac{1}{2}[w_0(x)+w_0(1-x)]=\frac{a_0}{2}+\sum_{n=1}^{+\infty} a_n T_n[1-8x(1-x)] \tag{11a}$$

$$\tfrac{1}{2}[w_0(x)-w_0(1-x)]=4[x(1-x)]^{1/2}(1-2x)\sum_{n=1}^{+\infty} b_n U_{n-1}[1-8x(1-x)]. \tag{11b}$$

If $w_0(x)$ is to play the role of probability density distribution, it should be normalised to unity in $[0, 1]$. On account of the uniformity of the convergence of (9) in the intervals $]0, x_1[, ]x_1, x_2[, \ldots, ]x_q, 1[$, we have

$$\begin{aligned}\int_0^1 w_0(x)\,\mathrm{d}x &= \frac{a_0}{2}+\sum_{n=1}^{+\infty} a_n \int_0^1 T_n[1-8x(1-x)]\,\mathrm{d}x\\ &=\frac{a_0}{2}+\sum_{n=1}^{+\infty} a_n \int_0^{\pi/2} \cos 4n\theta \sin 2\theta\,\mathrm{d}\theta\\ &=\frac{a_0}{2}-\sum_{n=1}^{+\infty}\frac{a_n}{4n^2-1}=1.\end{aligned} \tag{12}$$

It should be mentioned, however, that under the assumptions made, $w_0(x)$ is not necessarily positive-semidefinite on $[0, 1]$. Thus, we do not only consider functions representing probability density distributions, but also functions which may change sign a (finite) number of times in $[0, 1]$ provided that they can be normalised to unity.

Thus, starting from $w_0(x)$ represented by (8*a*) or (9) under the conditions mentioned, we construct the sequence

$$w_1(x), w_2(x), \ldots, w_m(x), \ldots \tag{13}$$

using the algorithm

$$w_{t+1}(x)=\frac{1}{4(1-x)^{1/2}}\left[w_t\left(\frac{1-(1-x)^{1/2}}{2}\right)+w_t\left(\frac{1+(1-x)^{1/2}}{2}\right)\right] \qquad \forall t \in \mathbb{N}. \tag{14}$$

First of all, we notice that the part of $w_0(x)$ which is antisymmetric with respect to $x=\frac{1}{2}$, namely (11*b*), yields no contribution to $w_1(x)$ and therefore also none to the other functions in (13). Hence, using $w_0(x)$ as the initial function is identical to starting the sequence (13) from

$$W_0^{\mathrm{sym}}(x)\equiv\tfrac{1}{2}[w_0(x)+w_0(1-x)]=\frac{a_0}{2}+\sum_{n=1}^{+\infty} a_n T_n[1-8x(1-x)]. \tag{15}$$

Next, replacing $x$ by $[1-(1-x)^{1/2}]/2$ in (15), we find

$$W_0^{\mathrm{sym}}\left(\frac{1-(1-x)^{1/2}}{2}\right)=\frac{a_0}{2}+\sum_{n=1}^{+\infty} a_n T_n(1-2x) \tag{16}$$

and the equality remains valid in $[0, 1]$ since we have simply carried out a bijective mapping

$$x=\tfrac{1}{2}[1-(1-y)^{1/2}] \qquad y=4x(1-x) \qquad y\in[0,1] \qquad x\in[0,\tfrac{1}{2}] \tag{17}$$

replacing $y$ by $x$ in the end. This can be done similarly for the replacement of $x$ by $[1+(1-x)^{1/2}]/2$ and hence

$$\begin{aligned} w_1(x) &= \frac{1}{4(1-x)^{1/2}}\left[w_0\left(\frac{1-(1-x)^{1/2}}{2}\right)+w_0\left(\frac{1+(1-x)^{1/2}}{2}\right)\right] \\ &= \frac{1}{4(1-x)^{1/2}}\left[W_0^{\mathrm{sym}}\left(\frac{1-(1-x)^{1/2}}{2}\right)+W_0^{\mathrm{sym}}\left(\frac{1+(1-x)^{1/2}}{2}\right)\right] \\ &= \frac{1}{2(1-x)^{1/2}}\left(\frac{a_0}{2}+\sum_{n=1}^{+\infty} a_n T_n(1-2x)\right). \end{aligned} \tag{18}$$

The same procedure may be repeated an arbitrary (finite) number of times, without any need for verifying at each step that the series of transformed terms is still convergent and has the corresponding transformed $w$ function as its sum, since the algorithm is based on carrying out the same changes of variables on both sides of an equality by means of bijective mappings. Consequently, after $m$ transformations, we have

$$w_m(x) = \left[\frac{a_0}{2}\right]_m + \sum_{n=1}^{+\infty} a_n[T_n[1-8x(1-x)]]_m \qquad m\in\mathbb{N}_0 \tag{19}$$

where $[\ldots]_m$ symbolises the result of applying the relation (14) $m$ times, namely with $t=0,1,\ldots,m-1$, successively. According to Nandakumaran (1985), we can write directly

$$\begin{aligned} \left[\frac{a_0}{2}\right]_m &= \frac{a_0}{4[x(1-x)]^{1/2}}\left[\prod_{j=1}^{m-1}\cos\left(\frac{\pi}{2^{j+1}}\right)\right]\sin\left[\frac{\theta}{2^{m-1}}+\frac{\pi}{2}\left(1-\frac{1}{2^{m-1}}\right)\right] \\ &= \frac{a_0\cos[(\pi-2\theta)/2^m]}{2^{m+1}[x(1-x)]^{1/2}\sin(\pi/2^m)} \qquad m\in\mathbb{N}_0 \end{aligned} \tag{20}$$

in which

$$\theta = \sin^{-1}(x^{1/2}) \qquad \forall x\in[0,1]. \tag{21}$$

The remaining task consists in finding the $m$th transform of $T_n[1-8x(1-x)]$ whereby $n\in\mathbb{N}_0$, making use of Nandakumaran's result for the transforms of the positive integer powers of $x(1-x)$. Introducing the shorthand notation

$$\begin{aligned} F_m(\theta,2p+1) &= \left[\prod_{j=1}^{m-1}\cos\left(\frac{(2p+1)\pi}{2^{j+1}}\right)\right]\sin\left[\frac{(2p+1)\theta}{2^{m-1}}+\frac{(2p+1)\pi}{2}\left(1-\frac{1}{2^{m-1}}\right)\right] \\ &= \frac{\cos[(2p+1)(\pi-2\theta)/2^m]}{2^{m-1}\sin[(2p+1)\pi/2^m]} \qquad \forall m\in\mathbb{N}_0 \qquad \forall p\in\mathbb{N} \end{aligned} \tag{22}$$

Nandakumaran's formula for the $m$th transform of $x^n(1-x)^n$ is

$$[x^n(1-x)^n]_m = \frac{(-1)^n}{2^{4n+1}[x(1-x)]^{1/2}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}F_m(\theta,2n-2k+1) \tag{23}$$

still with the relation (21) connecting $\theta$ and $x$. The transforms of $x^n(1-x)^n$ involve $(n+1)$ expressions of the kind (22). In the appendix, it is proven for $n\geqslant 1$ that the transforms of $T_n[1-8x(1-x)]$ involve only two $F_m$ functions:

$$[T_n[1-8x(1-x)]]_m = \frac{1}{4[x(1-x)]^{1/2}}[F_m(\theta,2n+1)-F_m(\theta,2n-1)] \tag{24}$$

with $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$, $\forall x \in [0, 1]$, $\theta = \sin^{-1}(x^{1/2})$. Consequently,

$$w_m(x) = \frac{1}{2^{m+1}[x(1-x)]^{1/2}} \left[ a_0 \frac{\cos[(\pi - 2\theta)/2^m]}{\sin(\pi/2^m)} + \sum_{n=1}^{+\infty} a_n \left( \frac{\cos[(2n+1)(\pi-2\theta)/2^m]}{\sin[(2n+1)\pi/2^m]} - \frac{\cos[(2n-1)(\pi-2\theta)/2^m]}{\sin[(2n-1)\pi/2^m]} \right) \right] \qquad \forall m \in \mathbb{N}_0 \tag{25}$$

where

$$2\theta = \cos^{-1}(1-2x).$$

When one lets $m$ tend to $+\infty$ in every term on the right-hand side, one obtains the series

$$\frac{1}{\pi[x(1-x)]^{1/2}} \left( \frac{a_0}{2} - \sum_{n=1}^{+\infty} \frac{a_n}{4n^2-1} \right) \tag{26}$$

and by virtue of the normalisation condition (12), the sum of this series is equal to the invariant density distribution (3). However, how far one can claim that the limit of $w_m(x)$ when $m \to +\infty$ is equal to the sum of the series (26) constructed by means of the limits of all terms in the right-hand side of (25) is an open question. A sufficient condition which will create uniform convergence, but which will unfortunately entail an additional restriction on the symmetric part of $w_0(x)$, can be obtained as follows.

From the first expression of $F_m(\theta, 2p+1)$, it is clear that

$$|F_m(\theta, 2p+1)| \leqslant 1 \qquad \forall \theta \in [0, \pi/2] \qquad \forall m \in \mathbb{N}_0 \qquad \forall p \in \mathbb{N}. \tag{27}$$

Hence, from (25) we deduce

$$\left| [x(1-x)]^{1/2} w_m(x) - \tfrac{1}{4} \left( a_0 F_m(\theta, 1) + \sum_{n=1}^{l} a_n [F_m(\theta, 2n+1) - F_m(\theta, 2n-1)] \right) \right| \leqslant \tfrac{1}{2} \sum_{n=l+1}^{+\infty} |a_n| \qquad \forall m \in \mathbb{N}_0 \qquad l \in \mathbb{N}_0. \tag{28}$$

If the numerical series

$$\frac{|a_0|}{2} + \sum_{n=1}^{+\infty} |a_n| \tag{29}$$

constructed with the coefficients of the expansion (15) is assumed to be convergent, then for any given arbitrarily small positive number $\varepsilon$, there exists a positive bound $N_\varepsilon$, independent of $m$, such that the left-hand side of (28) is smaller than $\varepsilon$ as soon as $l > N_\varepsilon$ for any $m \in \mathbb{N}_0$. This is uniform convergence with respect to $m$ and consequently, we have

$$\left| [x(1-x)]^{1/2} \lim_{m \to +\infty} w_m(x) - \frac{1}{\pi} \left( \frac{a_0}{2} - \sum_{n=1}^{l} \frac{a_n}{4n^2-1} \right) \right| \leqslant \varepsilon \qquad l > N_\varepsilon \tag{30}$$

which proves that

$$\lim_{m \to +\infty} w_n(x) = \frac{1}{\pi[x(1-x)]^{1/2}} \left( \frac{a_0}{2} - \sum_{n=1}^{+\infty} \frac{a_n}{4n^2-1} \right) = \frac{1}{\pi[x(1-x)]^{1/2}} \tag{31}$$

again by virtue of (12). But the convergence of (29) is precisely the condition of Weierstrass' criterion ensuring the absolute and uniform convergence of

$$\frac{a_0}{2}+\sum_{n=1}^{+\infty} a_n \cos 4n\theta$$

in $0\leq\theta\leq\pi/2$ and this, in turn, entails the continuity of

$$\tfrac{1}{2}\{g(\theta)+g[(\pi/2)-\theta]\}$$

in $[0, \pi/2]$ or, equivalently, the continuity of $W_0^{\mathrm{sym}}(x)$ in $[0, 1]$. The conclusion is as follows.

If, in $[0, 1]$, the initial function $w_0(x)$ is the sum of an arbitrary real, single-valued, bounded function antisymmetric with respect to $x=\frac{1}{2}$, and a normalised, real, single-valued, continuous function symmetric with respect to $x=\frac{1}{2}$, having an absolutely convergent series expansion of the type (11$a$) in $[0, 1]$, then under the algorithm (14) the following equality holds:

$$\lim_{m\to+\infty} w_m(x)=\frac{1}{\pi[x(1-x)]^{1/2}}. \tag{32}$$

The convergence of (29) is a sufficient, but not a necessary, condition because taking the limit on both sides of (25) when $m\to+\infty$ could lead to an equality even if (29) is not convergent. Hence, starting the process of successive transformations from a function $w_0(x)$ in which the part $W_0^{\mathrm{sym}}(x)$ involves a number of finite jumps may also yield (32) and it is my conjecture that it does, but at the moment I can give no proof of this conjecture.

I wish to make three final remarks.

(i) The $m$th transform of $T_n[1-8x(1-x)]$ under (14) is expressed in terms of a difference of two $F_m$ functions, according to (24). The polynomial functions whose $m$th transform involves only one $F_m$ can easily be obtained by combining (24) rewritten as

$$[\cos 4n\theta]_m=\frac{1}{4[x(1-x)]^{1/2}}[F_m(\theta,2n+1)-F_m(\theta,2n-1)] \qquad \forall n\in\mathbb{N}_0$$

and

$$[1]_m=\frac{1}{2[x(1-x)]^{1/2}}F_m(\theta,1) \qquad \forall\theta\in[0,\pi/2]$$

which follows from (23) for $n=0$. In this way, we have

$$\frac{1}{4[x(1-x)]^{1/2}}F_m(\theta,2n+1)=\tfrac{1}{2}[1]_m+\sum_{k=1}^{n}[\cos 4k\theta]_m$$
$$=\tfrac{1}{2}\left[1+2\sum_{k=1}^{n}\cos 4k\theta\right]_m=\tfrac{1}{2}\left[\frac{\sin(4n+2)\theta}{\sin 2\theta}\right]_m=\tfrac{1}{2}[U_{2n}(1-2x)]_m.$$

Hence

$$[U_{2n}(1-2x)]_m=\frac{F_m(\theta,2n+1)}{2[x(1-x)]^{1/2}}=\frac{\cos[(2n+1)(\pi-2\theta)/2^m]}{2^m[x(1-x)]^{1/2}\sin[(2n+1)\pi/2^m]} \tag{33}$$

valid for any $m\in\mathbb{N}_0$, any $n\in\mathbb{N}$ and with $\theta$ and $x$ still connected by (21) or equivalently by $2\theta=\cos^{-1}(1-2x)$.

(ii) Nandakumaran's formula

$$[r_{s_1} r_{s_2} \ldots r_{s_t}(\sin^2\theta)]^{n+1/2} = (\sin\Phi)^{2n+1} \qquad 0 \leqslant \theta \leqslant \pi/2 \tag{34}$$

where

$$r_s(x) = \tfrac{1}{2}[1 + s(1-x)^{1/2}] \qquad s = \pm 1$$

$$\Phi = \frac{\theta}{2^{t-1}} + \sum_{j=1}^{t-1} \tfrac{1}{2}(1+s_j)\pi/2^j$$

not only contains a printing error, namely the subscript $s_t$ in (34) should be replaced by $s_{t-1}$, but it is also incorrect from $t=3$ onward. Leaving out the irrelevant exponents in (34), momentarily, I assert that writing

$$r_{s_1} r_{s_2} \ldots r_{s_{t-1}}(\sin^2\theta) = \sin^2\left(\frac{\theta}{2^{t-1}} + \sum_{j=1}^{t-1} (1+s_j)\frac{\pi}{2^{j+1}}\right) \qquad 0 \leqslant \theta \leqslant \pi/2 \tag{35}$$

is erroneous for $t \geqslant 3$. This can easily be verified in the simplest case, namely $t=3$, when one compares

$$r_{s_1} r_{s_2}(\sin^2\theta) = \tfrac{1}{2} + \tfrac{1}{2}s_1[\tfrac{1}{2} - \tfrac{1}{2}s_2(1-\sin^2\theta)^{1/2}]^{1/2} \tag{36}$$

with

$$\sin^2\Phi = \sin^2[\tfrac{1}{4}\theta + (1+s_1)\tfrac{1}{4}\pi + (1+s_2)\tfrac{1}{8}\pi]. \tag{37}$$

For $(s_1, s_2) = (1, -1)$ and $(-1, -1)$, the two right-hand sides are equal, but for $(s_1, s_2) = (1, 1)$ and $(-1, 1)$, they are not and a permutation can be noted. The proof of a formula of type (35) is based upon the fact that every application of the operator $r_s$ transforms the square of a sine into the square of another sine, but the error which has been committed is that the sign of the intermediary cosine was not properly taken into account. Indeed,

$$\tfrac{1}{2} + \tfrac{1}{2}s(1-\sin^2\alpha)^{1/2} = \tfrac{1}{2} + \tfrac{1}{2}s\cos\alpha$$

holds provided $0 \leqslant \alpha < \pi/2$, whereas

$$\tfrac{1}{2} + \tfrac{1}{2}s(1-\sin^2\alpha)^{1/2} = \tfrac{1}{2} - \tfrac{1}{2}s\cos\alpha$$

should be written when $\pi/2 < \alpha \leqslant \pi$, the two formulae being valid for $\alpha = \pi/2$. Hence, by virtue of

$$r_-(\sin^2\theta) = \sin^2\tfrac{1}{2}\theta \qquad r_+(\sin^2\theta) = \cos^2\tfrac{1}{2}\theta = \sin^2(\tfrac{1}{2}\theta + \tfrac{1}{2}\pi) \qquad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi$$

we find in the case of (36):

$$r_{s_1} r_-(\sin^2\theta) = \tfrac{1}{2} + \tfrac{1}{2}s_1(1-\sin^2\tfrac{1}{2}\theta)^{1/2} = \tfrac{1}{2} + \tfrac{1}{2}s_1\cos\tfrac{1}{2}\theta = \sin^2[\tfrac{1}{4}\theta + (1+s_1)\tfrac{1}{4}\pi]$$

in agreement with (37), and

$$\begin{aligned} r_{s_1} r_+(\sin^2\theta) &= \tfrac{1}{2} + \tfrac{1}{2}s_1[1-\sin^2(\tfrac{1}{2}\theta + \tfrac{1}{2}\pi)]^{1/2} = \tfrac{1}{2} - \tfrac{1}{2}s_1\cos(\tfrac{1}{2}\theta + \tfrac{1}{2}\pi) \\ &= \sin^2[\tfrac{1}{4}\theta + (1-s_1)\tfrac{1}{4}\pi + \tfrac{1}{4}\pi] \end{aligned}$$

which does not agree with (37). These results may be united by writing

$$r_{s_1} r_{s_2}(\sin^2\theta) = \sin^2[\tfrac{1}{4}\theta + (1-s_1 s_2)\tfrac{1}{4}\pi + (1+s_2)\tfrac{1}{8}\pi].$$

The generalisation of this formula, i.e.

$$r_{s_1}r_{s_2}\ldots r_{s_n}(\sin^2\theta)=\sin^2\left(\frac{\theta}{2^n}+\sum_{j=1}^{n}\left[1+(-1)^{n-j}s_js_{j+1}\ldots s_n\right]\frac{\pi}{2^{j+1}}\right)\qquad \forall n\in\mathbb{N}_0 \tag{38}$$

can be proven by complete induction. Still assuming $0\leqslant\theta\leqslant\pi/2$, we have

$$r_{s_1}r_{s_2}\ldots r_{s_{n+1}}(\sin^2\theta)=\frac{1}{2}+\frac{s_1}{2}\left[\cos^2\left(\frac{\theta}{2^n}+\sum_{j=1}^{n}\left[1+(-1)^{n-j}s_{j+1}s_{j+2}\ldots s_{n+1}\right]\frac{\pi}{2^{j+1}}\right)\right]^{1/2}$$

where the $\cos^2$ argument belongs to

$$[0,\pi/2[\qquad \text{when } s_2s_3\ldots s_{n+1}=(-1)^n$$

$$[\pi/2,\pi[\qquad \text{when } s_2s_3\ldots s_{n+1}=(-1)^{n+1}.$$

Consequently,

$$\begin{aligned}
&r_{s_1}r_{s_2}\ldots r_{s_{n+1}}(\sin^2\theta)\\
&=\tfrac{1}{2}+\tfrac{1}{2}(-1)^n s_1s_2\ldots s_{n+1}\cos\left(\frac{\theta}{2^n}+\sum_{j=2}^{n+1}\left[1+(-1)^{n-j+1}s_js_{j+1}\ldots s_{n+1}\right]\frac{\pi}{2^j}\right)\\
&=\begin{cases}\sin^2\left(\dfrac{\theta}{2^{n+1}}+\displaystyle\sum_{j=2}^{n+1}\left[1+(-1)^{n-j+1}s_js_{j+1}\ldots s_{n+1}\right]\dfrac{\pi}{2^{j+1}}\right) & \text{when } s_1s_2\ldots s_{n+1}=(-1)^{n+1}\\[2ex]
\sin^2\left(\dfrac{\theta}{2^{n+1}}+\dfrac{\pi}{2}+\displaystyle\sum_{j=2}^{n+1}\left[1+(-1)^{n-j+1}s_js_{j+1}\ldots s_{n+1}\right]\dfrac{\pi}{2^{j+1}}\right) & \text{when } s_1s_2\ldots s_{n+1}=(-1)^n\end{cases}\\
&=\sin^2\left(\frac{\theta}{2^{n+1}}+\left[1+(-1)^n s_1s_2\ldots s_{n+1}\right]\tfrac{1}{4}\pi\right.\\
&\qquad\left.+\sum_{j=2}^{n+1}\left[1+(-1)^{n-j+1}s_js_{j-1}\ldots s_{n+1}\right]\frac{\pi}{2^{j+1}}\right)
\end{aligned}$$

which is indeed the right-hand side of (38) after replacement of $n$ by $n+1$.

If one wishes to correct (35) while keeping the right-hand side unmodified, our result (38) leads to the following formula:

$$r_{u_1}r_{u_2}\ldots r_{u_{t-2}}r_{s_{t-1}}(\sin^2\theta)=\sin^2\left(\frac{\theta}{2^{t-1}}+\sum_{j=1}^{t-1}(1+s_j)\frac{\pi}{2^{j+1}}\right) \tag{39}$$

where

$$u_j=-\frac{s_j}{s_{j+1}}\qquad \forall j\in\{1,2,\ldots,t-2\}.$$

In conclusion, the formulae (12) and (13) in Nandakumaran (1985) should read for $t\geqslant 2$:

$$[r_{s_1}r_{s_2}\ldots r_{s_{t-1}}(\sin^2\theta)]^{n+1/2}=(\sin\Phi)^{2n+1} \tag{40}$$

where

$$\Phi=\frac{\theta}{2^{t-1}}+\sum_{j=1}^{t-1}\left[1+(-1)^{t-j-1}s_js_{j+1}\ldots s_{t-1}\right]\frac{\pi}{2^{j+1}}. \tag{41}$$

Despite the error in (12) and (13), the final results of Nandakumaran (from (15) onward) are correct thanks to the fact that the final operation in his paper consists in summing with respect to $s_1, s_2, \ldots, s_{t-1}$, each $s$ taking on the values $+1$ and $-1$. Considering that (38) and (39) are equivalent (with $n = t-1$), and that when the group of variables $s_1, s_2, \ldots, s_{t-1}$ runs over the set of all permutations with repetition of the elements $+1$ and $-1$ taken $t-1$ at a time in (39), the group of variables $u_1, u_2, \ldots, u_{t-2}, s_{t-1}$ does the same, it is clear that the error disappears in the course of the summation process.

(iii) Near the end of his letter, Nandakumaran makes use of the equality

$$\frac{(-1)^n}{B(n+1, n+1)} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \frac{1}{2n-2k+1} = 2^{4n} \qquad \forall n \in \mathbb{N} \tag{42}$$

which he mentions without proof as a generalisation of a few special cases. By coincidence, I was recently confronted with the same equality in research work of an entirely different nature. The proof of (42) is given in appendix A of Grosjean (1986).

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## Appendix. Proof of (24)

If

$$T_n[1-8x(1-x)] = \sum_{j=0}^{n} C_j x^j (1-x)^j \qquad n \in \mathbb{N}_0 \tag{A1}$$

then, on account of (23),

$$\begin{aligned}
&[T_n[1-8x(1-x)]]_m \\
&\quad = \frac{1}{2[x(1-x)]^{1/2}} \sum_{j=0}^{n} (-1)^j \frac{C_j}{2^{4j}} \\
&\qquad \times \sum_{k=0}^{j} (-1)^k \binom{2j+1}{k} F_m(\theta, 2j-2k+1) \\
&\quad = \frac{1}{2[x(1-x)]^{1/2}} \sum_{s=0}^{n} (-1)^s F_m(\theta, 2s+1) \\
&\qquad \times \sum_{j=s}^{n} \binom{2j+1}{j-s} \frac{C_j}{2^{4j}} \qquad n \in \mathbb{N}_0.
\end{aligned} \tag{A2}$$

Substituting $x = \sin^2(\varphi/2)$ in (A1), we find

$$\cos 2n\varphi = \sum_{j=0}^{n} \frac{C_j}{2^{2j}} \sin^{2j} \varphi \qquad \forall \varphi \in [-\pi, \pi]$$

and therefore, when $s \in \{0, 1, \ldots, n\}$,

$$\int_0^{\pi/2} \cos 2n\varphi \sin \varphi \sin(2s+1)\varphi \, d\varphi = \sum_{j=0}^{n} \frac{C_j}{2^{2j}} \int_0^{\pi/2} \sin^{2j+1} \varphi \sin(2s+1)\varphi \, d\varphi. \tag{A3}$$

But

$$\int_0^{\pi/2} \sin^{2j+1}\varphi\,\sin(2s+1)\varphi\,d\varphi$$

$$=\frac{1}{(2i)^{2j+1}}\int_0^{\pi/2}(e^{i\varphi}-e^{-i\varphi})^{2j+1}\sin(2s+1)\varphi\,d\varphi$$

$$=\frac{(-1)^j}{2^{2j}}\sum_{p=0}^{j}(-1)^p\binom{2j+1}{p}\int_0^{\pi/2}\sin(2j-2p+1)\varphi\,\sin(2s+1)\varphi\,d\varphi$$

$$=\frac{(-1)^j\pi}{2^{2j+2}}\sum_{p=0}^{j}(-1)^p\binom{2j+1}{p}\delta_{j-p-s,0}$$

$$=\begin{cases}0 & \text{when } 0\leqslant j\leqslant s-1\\ \dfrac{(-1)^s\pi}{2^{2j+2}}\dbinom{2j+1}{j-s} & \text{when } j\geqslant s.\end{cases}$$

Consequently, (A3) yields

$$(-1)^s\tfrac{1}{4}\pi\sum_{j=s}^{n}\frac{C_j}{2^{4j}}\binom{2j+1}{j-s}=\int_0^{\pi/2}\cos 2n\varphi\,\sin\varphi\,\sin(2s+1)\varphi\,d\varphi$$

$$=\tfrac{1}{8}\pi(\delta_{n-s,0}-\delta_{n-s-1,0})$$

and so

$$[T_n[1-8x(1-x)]]_m=\frac{1}{4[x(1-x)]^{1/2}}\sum_{s=0}^{n}F_m(\theta,2s+1)(\delta_{n-s,0}-\delta_{n-s-1,0})$$

which proves (24) for any $n\in\mathbb{N}_0$.

## References

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